PLANE PROBLEM FOR AN ORTHOTROPIC BODY WITH A CRACK WHOSE LOWER SIDE IS REINFORCED BY AN ELASTIC MEMBRANE

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We study the stress state of an orthotropic plane with one linear defect whose lower side is reinforced by an elastic membrane. The Lekhnitskii potentials are constructed as solutions of the Riemann two-dimensional boundary-value problem. They are obtained in closed form. It is shown that the asymptotic behavior of stresses at the tips of the defect can have a singularity of any order from −1 to 0, depending on the stiffness of the membrane. The cases of low and high stiffness are considered separately.

Two types of singularities of loading factors are distinguished at the tips of linear defects in a deformable solid body. Under linear elastic constraints, singularities of the order of $r^{-1/2}$ are typical of an ideal crack and an infinitely thin stiff inclusion. [1]. Singularities of the orders of $r^{-1/4}$ and $r^{-3/4}$ are typical of the conditions of the primal mixed boundary-value problem (a stiff inclusion with a detached side) [2].

In the present paper, we consider a boundary-value problem for which the crack-tip stresses can have a singularity of any order from 0 to −1. It is shown, in particular, that asymmetric reinforcement of the crack side changes the stress-strain state qualitatively compared to the case of an ideal crack with a free side.

1. Boundary Conditions of the Problem. We consider an unbounded plane orthotropic medium with different and purely imaginary characteristic numbers ($s_k = im_k$, $k = 1,\ 2$). In this case, the matrix of elastic compliances a_{ij} [3] satisfies the condition $2\sqrt{a_{11}a_{22}} \neq 2a_{12}+a_{66}$, and the orthotropy directions coincide with the coordinate directions.

Stresses, strains, displacements, and other loading factors are expressed in terms of two analytical functions [$\Phi(z_1)$ and $\Psi(z_2)$] by the known general rules [3, 4].

We assume that the medium is weakened by a cut along the segment of the real axis $|x| < 1$. The lower side of the cut is continuously reinforced by an infinitely thin elastic membrane of constant stiffness. By definition, the membrane prevents elastically longitudinal tensile–compressive strains and does not resist bending and shear. In [5, 6], this membrane is called a "momentless elastic fiber."

At infinity the medium is loaded by uniform stresses σ_{ij}^0. The contour of the defect is free of external forces (see Fig. 1). In this case, the static equilibrium of the boundary elements of the solid body is ensured by conditions of the form

$$\sigma_{22}^{\pm}(x,0) = 0, \quad \sigma_{12}^{+}(x,0) = 0, \quad \sigma_{1\lambda}^{-}(x,0) \equiv \alpha\sigma_{12}^{-}(x,0) + \beta\sigma_{11}^{-}(x,0) = 0, \tag{1.1}$$

in which the direction cosines and related parameters are given by

$$\alpha = \frac{1}{\sqrt{1+c^2a_{11}^2}}, \quad \beta = \frac{ca_{11}}{\sqrt{1+c^2a_{11}^2}}, \quad \mu \equiv \alpha + i(m_1+m_2)\beta, \quad e^{2i\lambda} \equiv \frac{\bar{\mu}}{\mu}. \tag{1.2}$$

Here c is the tension–compression stiffness of the elastic fiber.

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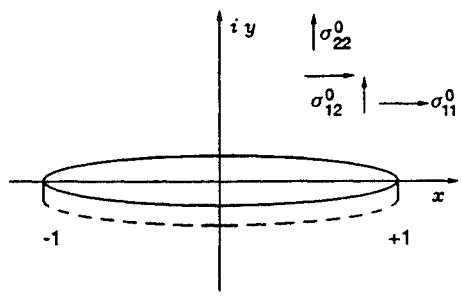

Fig. 1

2. Boundary-Value Problems. We shall seek the Lekhnitskii potentials in the form

$$\Phi(z_1) = \Phi_1(z_1) + i\Phi_2(z_1), \qquad \Psi(z_2) = \Psi_1(z_2) + i\Psi_2(z_2), \tag{2.1}$$

where the functions $\Phi_{1,2}$ and $\Psi_{1,2}$ are represented by Cauchy-type integrals with purely real densities. We describe briefly the procedure of constructing potentials (2.1).

The first pair of boundary conditions (1.1) leads to the following two Riemann boundary-value problems for functions (2.1):

$$\begin{gathered}\frac{1}{2}(\sigma_{22}^+ - \sigma_{22}^-) \equiv (\Phi_1^+ - \Phi_1^-) + (\Psi_1^+ - \Psi_1^-) = 0,\\ \frac{1}{2i}(\sigma_{22}^+ + \sigma_{22}^-) \equiv (\Phi_2^+ + \Phi_2^-) + (\Psi_2^+ + \Psi_2^-) - i\sigma_{22}^0 = 0.\end{gathered} \tag{2.2}$$

The solution of the boundary-value problems (2.2) is a matter of common knowledge [7] and one of the potentials to be eliminated. For definiteness, we assume that the second potential $\Psi(z_2)$ is eliminated. In this case, the second pair of boundary conditions (1.1) leads to the Riemann two-dimensional boundary-value problem for the first potential. A distinguishing feature of the second problem is that the matrix of its coefficients does not depend on spatial coordinates and can be brought to the diagonal form by rotation through an angle $\lambda/2$. Thus, we finally arrive at two Riemann independent boundary-value problems and an expression for the sought potential $\Phi(z_1)$:

$$\varphi_k^+ - (-1)^k e^{-i\lambda}\varphi_k^- = F_k, \qquad k = 1, 2; \tag{2.3}$$

$$\Phi \equiv \Phi_1 + i\Phi_2 = e^{i\lambda/2}(\varphi_1 + i\varphi_2); \tag{2.4}$$

$$\begin{aligned} F_1(x) &= \frac{ie^{-i\lambda/2}}{2(m_2 - m_1)}\left\{\frac{m_2\sigma_{22}^0 + \sigma_{11}^0}{m_2 + m_1}\sin\lambda + \sigma_{12}^0(1 + \cos\lambda)\right\} + \frac{im_2\sigma_{22}^0(1 - \cos\lambda)}{2(m_2 - m_1)}\frac{x}{\sqrt{1 - x^2}}e^{-i\lambda/2},\\ F_2(x) &= \frac{ie^{-i\lambda/2}}{2(m_2 - m_1)}\left\{\sigma_{12}^0(1 - \cos\lambda) - \frac{m_2\sigma_{22}^0 + \sigma_{11}^0}{m_2 + m_1}\sin\lambda\right\} + \frac{im_2\sigma_{22}^0(1 + \cos\lambda)}{2(m_2 - m_1)}e^{-i\lambda/2}\frac{x}{\sqrt{1 - x^2}}. \end{aligned} \tag{2.5}$$

Solutions of boundary-value problems (2.3) are sought by the general rules [7] and involve no difficulties. The only solution that is not bounded at both tips of the defect is obtained using the conditions of uniqueness of displacements and absence of the resultant force vector at the defect in the form indicated in [4]. We give the final result:

$$\begin{gathered}\varphi_k(z_1) = \frac{X_k(z_1)}{2\pi i}\int_{-1}^{+1}\frac{F_k(t)}{X_k^+(t)}\frac{dt}{t - z_1}, \qquad k = 1, 2,\\ X_1(z) = (z-1)^{-n-1/2}(z+1)^{n-1/2}, \quad X_2(z) = (z-1)^{-n}(z+1)^{n-1}, \quad n = \lambda/2\pi.\end{gathered} \tag{2.6}$$

Here $X_{1,2}(z_1)$ are solutions of the homogeneous problems (2.3)

The expressions for the second Lekhnitskii potential $\Psi(z_2)$ are constructed similarly to the procedure described above and have the form

$$\Psi(z_2) \equiv \Psi_1 + i\Psi_2 = e^{i\lambda/2}(\psi_1 + i\psi_2), \quad \psi_k(z_2) = \frac{X_k(z_2)}{2\pi i} \int_{-1}^{+1} \frac{F_k^*(t)}{X_k^+(t)} \frac{dt}{t - z_2}, \quad k = 1, 2, \tag{2.7}$$

where the functions $F_k^*(t)$ are defined by the expressions for the function $F_k(t)$ (2.5) in which the substitution $m_2 \leftrightarrow m_1$ is performed and $X_k(z)$ are given by formula (2.6). The similarity of formulas (2.4) and (2.7) is due to the symmetry of stresses about the complex potentials $\Phi(z_1)$ and $\Psi(z_2)$.

The set of equalities (2.3)–(2.7) is an integral form of the exact solution of the formulated problem (1.1).

3. Transformation of the Exact Solution. The integral terms in the exact solution can be calculated by extending the well-known method of calculation of Cauchy-type integrals, which is described in, e.g., [8]. The method is extended to integrals (2.6) and (2.7), whose values are sought in the form

$$\frac{1}{2\pi i} \int_{-1}^{+1} \frac{F(t)}{X(t)} \frac{dt}{t - z} = \omega\Big\{\frac{F(t)}{X(t)} - G(z)\Big\}, \tag{3.1}$$

where $G(z)$ is the sum of the main parts of the function F/X at its poles, and the constant ω is determined by comparison of the limiting values of the left and right sides of inequality (3.1) on the line of integration with allowance for the boundary-value problem for the function $X(z)$. In particular, for the function $F(t) = F_{1,2}(t)$ the following two integrals are of interest:

$$J_1(z,\lambda) \equiv \frac{X_1(z)}{2\pi i} \int_{-1}^{+1} \frac{1}{X_1^+(t)} \frac{dt}{t - z} = \frac{1}{1 + e^{-i\lambda}} \Big\{1 - \Big(z - \frac{\lambda}{\pi}\Big) X_1(z)\Big\},$$

$$J_2(z,\lambda) \equiv \frac{X_1(z)}{2\pi i} \int_{-1}^{+1} \frac{1}{X_1^+(t)} \frac{t}{\sqrt{1-t^2}} \frac{dt}{t - z} = \frac{1}{1 - e^{-i\lambda}} \Big\{\frac{z}{\sqrt{1-z^2}} + i\Big(z - \frac{\lambda}{\pi}\Big) X_1(z)\Big\}. \tag{3.2}$$

The expressions for similar integrals that contain $X_2(z)$ and enter into the functions $\varphi_2(z_1)$ and $\psi_2(z_2)$ are obtained from formulas (3.2) by replacing $\lambda \to \lambda - \pi$.

Finally, using equalities (2.4)–(2.6) and (3.2), we arrive at the exact value of the first Lekhnitskii potential in analytic form:

$$\begin{aligned}\Phi(z_1) = {} & \frac{i}{2(m_2 - m_1)} \Big\{\sigma_{12}^0(1 + \cos\lambda) + \frac{m_2\sigma_{22}^0 + \sigma_{11}^0}{m_2 + m_1} \sin\lambda\Big\} J_1(z_1, \lambda) \\ & + \frac{i m_2 \sigma_{22}^0}{2(m_2 - m_1)} \{(1 - \cos\lambda) J_2(z_1, \lambda) + (1 + \cos\lambda) J_2(z_1, \lambda - \pi)\} \\ & + \frac{i}{2(m_2 - m_1)} \Big\{\sigma_{12}^0(1 - \cos\lambda) - \frac{m_2\sigma_{22}^0 + \sigma_{11}^0}{m_2 + m_1} \sin\lambda\Big\} J_1(z_1, \lambda - \pi).\end{aligned} \tag{3.3}$$

The second potential $\Psi(z_2)$ in analytic form is obtained from relation (3.3) by performing the substitutions

$$z_1 \to z_2, \qquad m_2 \leftrightarrow m_1. \tag{3.4}$$

4. Case of an Isotropic Medium. The structure of potentials (3.3) admits direct passage to an isotropic medium by the scheme of [4]. After appropriate manipulations we obtain relations for the standard stress complexes:

$$\sigma_{11} + \sigma_{22} = \sigma_{11}^0 + \sigma_{22}^0 - 2\,\mathrm{Im}\Big\{\Big(\sigma_{12}^0(1 + \cos\lambda) + \frac{1}{2}(\sigma_{11}^0 + \sigma_{22}^0)\sin\lambda\Big) J_1(z,\lambda)$$

$$+\sigma_{22}^0((1-\cos\lambda)J_2(z,\lambda)+(1+\cos\lambda)J_2(z,\lambda-\pi))$$

$$+\Big(\sigma_{12}^0(1-\cos\lambda)-\frac{1}{2}(\sigma_{11}^0+\sigma_{22}^0)\sin\lambda\Big)J_1(z,\lambda-\pi)\Big\}, \tag{4.1}$$

$$\sigma_{22}-\sigma_{11}+2i\sigma_{12}=\sigma_{22}^0-\sigma_{11}^0+2i\sigma_{12}^0-2\mathrm{Re}\Big\{(2-i)\Big(\sigma_{12}^0(1+\cos\lambda)+\frac{1}{2}(\sigma_{11}^0+\sigma_{22}^0)\sin\lambda\Big)$$

$$\times J_1(z,\lambda)+(2-i)\Big(\sigma_{12}^0(1-\cos\lambda)-\frac{1}{2}(\sigma_{11}^0+\sigma_{22}^0)\sin\lambda\Big)J_1(z,\lambda-\pi)\Big\}.$$

5. Asymptotic Stress Relations. In the neighborhoods of the left and right tips of the defect, the complex potentials are described by different asymptotic relations. For example, at the right tip of the defect, the exact solution (3.3) is described by the asymptotic relation ($\zeta_1 = z_1 - 1$)

$$\Phi(z_1)=\frac{\sigma_{22}^0 m_2}{2\sqrt{2}(m_2-m_1)}\zeta_1^{-1/2}+c_1\zeta_1^{-n-1/2}+c_2\zeta_1^{-n}, \tag{5.1}$$

and at the left tip ($\zeta_1 = z_1 + 1$), it is described by

$$\Phi(z_1)=\frac{\sigma_{22}^0 m_2}{2\sqrt{2}(m_2-m_1)}\zeta_1^{-1/2}+c_3\zeta_1^{n-1}+c_4\zeta_1^{n-1/2}. \tag{5.2}$$

Here the constants $c_1,\ldots,c_4$ are calculated from the exact solution of (3.3). In this case, the asymptotic expressions for the other potential $\Psi(z_2)$ are obtained from formulas (5.1) and (5.2) using the substitution (3.4).

The first terms in the asymptotic relations (5.1) and (5.2) describe the usual state of ordinary separation, as in the case of an ideal crack [3]. The remaining terms describe the redistribution of the stress fields caused by the effect of the reinforcing membrane. The parameter n depends on the stiffness of the membrane via the angle λ (1.2) and varies in the range $0 < n \leqslant 1/4$, where the left value is typical of infinitely low stiffness ($c = \lambda = 0$) and the right value is typical of infinitely high stiffness ($c = \infty$ and $\lambda = \pi/2$). As a result, depending on the stiffness of the reinforcing membrane, singularities of any order from 0 to $(-1+\varepsilon)$, where $\varepsilon \to 0$, can arise at the tips of the defect. For low stiffness, higher-order singularities arise $(n-1)$ in the vicinity of the left tip of the defect.

When the stiffness of the reinforcing membrane is infinitely high, the asymptotic relations of the potentials have singular terms of the same order at both tips of the defect: $\zeta_1^{-1/2}$, $\zeta_1^{-1/4}$, and $\zeta_1^{-3/4}$. It is interesting to compare this case with the primal mixed problem of crack theory [2] and symmetric reinforcement of both sides of the defect by a stiff membrane. The primal mixed problem is characterized by stress oscillations near the tips of the defect and the associated uncertainty of the stress-intensity coefficients in asymptotic terms of the orders of $-1/4$ and $-3/4$. In the problem considered here with asymptotic relations (5.1) and (5.2), the intensity coefficients $c_1,\ldots,c_4$ are exactly known constants, and the stresses depend monotonically on the polar radius and do not have oscillating components.

In symmetric reinforcement of the sides of the defect, the stresses have a characteristic singularity of the order of $-1/2$ and an angular distribution that is not typical of cracks [9]. The asymmetry of the boundary conditions due to one-sided reinforcement changes qualitatively the stress-distribution pattern compared to the symmetric case. In this connection, we consider the limit of the infinitely small stiffness of the reinforcing fiber for which the singularity order in the asymptotic relations (5.2) is the highest. In the limit $n \to 0$, the first potential (3.3) has the following structure of the main singular terms:

$$\Phi(z_1)=\frac{m_2\sigma_{22}^0-i\sigma_{12}^0}{2(m_2-m_1)}\Big\{\frac{z_1}{\sqrt{z_1^2-1}}-1\Big\}+n\frac{\sigma_{11}^0+m_2\sigma_{22}^0}{2(m_2^2-m_1^2)}\Big\{\Big(\frac{z_1-1}{z_1+1}\Big)^{1-n}-1\Big\}+O(\lambda). \tag{6.1}$$

The corresponding relation for the second potential $\Psi(z_2)$ is obtained as before from formula (6.1) by substitutions (3.4). The first term is an exact solution for an ideal crack [3]. The last terms, vanishing as $n \to 0$, determine the stress redistribution under the effect of the reinforced edge. The factor responsible for the stress redistribution is that the edge is kept from rotations and dilatations by internal forces whose intensity is known to be calculated from the values of $\sigma_{22}^{-}(x,0)$ and $\sigma_{12}^{-}(x,0)$. Taking into account these factors, it is not difficult to write a solution for the case where the edge is free of internal forces, and, hence, the torsion moment and the moment of dilation at the edge are equal to zero. When just one of the indicated moments is equal to zero, solutions are constructed in a similar manner. All of them are similar in properties to the above potentials.

Conclusions. (1) We considered a mixed elastostatic problem of the deformation of a homogeneous plane medium weakened by a cut whose lower side is reinforced by an elastic membrane (a momentless elastic fiber). Exact solutions are constructed for the orthotropic and isotropic cases.

(2) The asymptotic expressions for the Lekhnitskii potentials show that, depending on the stiffness of the reinforcing membrane, stress singularities of any order from 0 to $(-1+\varepsilon)$, where $\varepsilon \to 0$, can arise at the tips of the defect. When the stiffness is small, a high-order singularity occurs at the left tip of the defect, but the corresponding intensity coefficient tends to zero.

(3) In contrast to the primal mixed problem, the constructed asymptotic relations for the stress fields depend monotonically on the polar radius, and the intensity coefficients are determined exactly and do not have oscillating components.

(4) Comparison with the case of symmetric reinforcement of both crack sides shows that the asymmetry of the boundary conditions changes the stress-distribution pattern qualitatively.

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